

Phenomenological analysis of properties of the RH Majorana neutrino in the seesaw mechanism

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Abstract

As an extension of our previous work in the seesaw mechanism, we analyze the influence of U_{e3} on the properties (masses and mixing) of the RH Majorana neutrinos in three flavors. The quasidegenerate light neutrinos case is also considered. Assuming the hierarchical Dirac neutrino masses, we find the heavy Majorana neutrino mass spectrum is either hierarchical or partial degenerate if θ_{23}^ν is large. We show that degenerate RH Majorana masses correspond to maximal RH mixing angle while hierarchical ones correspond to the RH mixing angles which scale linearly with the mass ratios of the Dirac neutrino masses. An interesting analogue with the behavior of the matter-

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enhanced neutrino conversion is also presented.

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I. INTRODUCTION

By adding the heavy right-handed (RH) neutrinos ν_R , the seesaw mechanism [1] provides a very natural and attractive explanation of the smallness of the neutrino masses compared to the masses of the charged fermions. In the seesaw mechanism the mass matrix of the left-handed (LH) neutrino has the following form [2]:

$$m_\nu = m_D M^{-1} m_D^T, \quad (1)$$

where m_D is the Dirac mass matrix and M is the Majorana mass matrix for the right-handed neutrino components. According to some kind of quark-lepton analogy suggested in Grand Unified Theories (GUTs), the structure of the Dirac mass matrix m_D are similar to that in the quark sector [2]. However, the scale of the RH Majorana neutrino masses is not precisely known. In various theoretical model, ν_R can be the unification scale ($\sim 10^{16}$ GeV) or the intermediate scale ($\sim 10^9 - 10^{13}$ GeV) [3]. To understand possible unification of particles and interactions, it is crucial to know if this scale associated with some new physics and at what energy it eventually happens [3].

The seesaw mechanism can give us hints about the neutrino properties in two aspects [1,4]. The first is fixing M by some ansatz and predicting masses and mixing of the light neutrinos. With the increasing of the data from the low neutrino experiments, it becomes pressing and practical to determine the structure of the RH Majorana neutrinos from the light neutrino masses and mixing implied by experiments [3–5].

The knowledge about neutrino masses and mixing comes mainly from three kinds of neutrino oscillation experiments: the solar [6] and atmospheric [7] neutrino deficits, LSND reactor experiments [8]. The observation of the CHOOZ Collaboration [9] implied a small U_{e3} [10]

$$|U_{e3}| = \sin \theta_{13}^\nu \leq 0.13 - 0.23 \quad (2)$$

and so the atmospheric neutrino oscillation $\nu_\mu \leftrightarrow \nu_\tau$ decouples approximately from the solar neutrino oscillation $\nu_e \leftrightarrow \nu_\mu$. Analysis of the atmospheric neutrino deficit observed by the Super-Kamiokande Collaboration yield the mass-squared difference [7]

$$\Delta m_{23}^2 = 5.9 \times 10^{-3} \text{ eV}^2 \quad (3)$$

with the almost maximal mixing $\sin^2 2\theta_{23}^\nu = 1$ [7]. In contrast, there exists two different oscillation mechanism yielding four possible solutions [11] to the solar neutrino problem: "Just-so" mechanism (or VO, i.e. the long wavelength vacuum oscillations) with

$$(\Delta m_{12}^2, \sin^2 2\theta_{12}^\nu) = (6.5 \times 10^{-11} \text{ eV}^2, 0.75) \quad (4)$$

and the Mikheyev-Smirnov-Wolfenstein (MSW) mechanism [12] (the matter-enhanced oscillation) with

$$\begin{aligned} (\Delta m_{12}^2, \sin^2 2\theta_{12}^\nu) &= (1.8 \times 10^{-5} \text{ eV}^2, 0.76) && \text{(LMA)}, \\ &= (7.9 \times 10^{-8} \text{ eV}^2, 0.96) && \text{(LOW)}, \\ &= (5.4 \times 10^{-6} \text{ eV}^2, 6.0 \times 10^{-3}) && \text{(SMA)}. \end{aligned} \quad (5)$$

Here LMA (SMA) refers to large (small) mixing angle and LOW stands for low mass or possibility. All the above we take are the best-fit values. For the convenience of late discussion, we present here the regions of the mass-squared differences Δm_{12}^2 from Bahcall and Δm_{23}^2 obtained from SK:

$$\begin{aligned} \Delta m_{12}^2: & 4 \times 10^{-12} \sim 6 \times 10^{-9} \text{ eV}^2 && \text{(VO)}, \\ & 6 \times 10^{-6} \sim 3 \times 10^{-4} \text{ eV}^2 && \text{(LMA)}, \\ & 3 \times 10^{-8} \sim 2 \times 10^{-7} \text{ eV}^2 && \text{(LOW)}, \\ & 4 \times 10^{-6} \sim 1 \times 10^{-5} \text{ eV}^2 && \text{(SMA)}, \\ \Delta m_{23}^2: & 1 \times 10^{-3} \sim 1 \times 10^{-1} \text{ eV}^2 && \text{(From SK)}. \end{aligned} \quad (6)$$

Another neutrino oscillation experiment, LSND, indicates the mass-squared difference

$$\Delta m_{LSND}^2 \sim 1 \text{ eV}^2 \quad (7)$$

with the mixing angle $\sin^2 2\theta_{LSND}^\nu \sim 10^{-3} - 10^{-2}$. Four neutrinos are needed to accommodate all the three mass-squared differences. However, the LSND results were not confirmed by the recent KARMEN experiment and we will just set it aside [13].

In the basis that the Dirac mass matrix of charged leptons is diagonal, the seesaw matrix [2], S , can be written as:

$$S = D_L^T U. \quad (8)$$

Here D_L is just U_0 in Ref. [5] which is the left-handed transformation to diagonalize m_D . We will ignore the CP-violating effect (i.e. all the mixing matrices entered in the seesaw mechanism are real). It is convenient to set $D_L = I$ and include its influence in the structure of U . In this paper we will consider how a nonzero U_{e3} will affect the RH neutrino masses

and mixing as an extension of our previous analysis [5]. There are three reason for us to devote to this topic: one is that θ_{13}^ν can be comparable with and can even be far larger than θ_{12}^ν in the allowed region of the small mixing MSW effect. The other is that S contains the contribution from D_L^T . If assuming m_D has the same structure, not only the mass scale but also the mixing matrix, with that in quark sector, one has [3]

$$D_L = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & \lambda^4 \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & \lambda^2 \\ \lambda^3 - \lambda^4 & -\lambda^2 & 1 - \frac{1}{2}\lambda^2 \end{pmatrix} \quad (9)$$

with $\lambda \approx 0.22$. Then S_{12} and S_{13} have the following values:

$$S_{12} = U_{e2} - \lambda U_{\mu 2} - \frac{1}{2}\lambda^2 U_{e2} + (\lambda^3 - \lambda^4) U_{\tau 2}, \quad (10)$$

$$S_{13} = U_{e3} - \lambda U_{\mu 3} - \frac{1}{2}\lambda^2 U_{e3} + (\lambda^3 - \lambda^4) U_{\tau 3}. \quad (11)$$

Since $\lambda \gg U_{e2} \sim 10^{-2}$ in the SMA region, one can see that $S_{12} \sim S_{13} \sim -\lambda$ even when $\theta_{13}^\nu = 0$. The third is that, from Eq. (15) and Eq. (17) (see in Sec. II), the coefficient of U_{e3} in A is the maximal one. Moreover, from Eq. (16) and Eq. (18), the coefficient of $U_{\tau 1}$ in B is also the maximal one. So the scale and structure of the Majorana neutrino may be sensitive, particularly in the SMA region i.e. when θ_{12}^ν is small, to the value of U_{e3} and $U_{\tau 1}$, that is, to the values of θ_{12}^ν and θ_{13}^ν .

In section II, the quasidegenerate mass case is briefly discussed. In section III, the hierarchical mass case is studied in two possibilities according to whether θ_{12}^ν is large or small. Finally, in section V we summary our main results.

II. QUASIDEGENERATE SPECTRUM

In Ref. [5] we have obtained the three eigenvalues of M which can be expressed as

$$M_1 = F\overline{M}_1, \quad M_2 = F\overline{M}_2, \quad M_3 = F\overline{M}_3, \quad (12)$$

and the eigenvectors of M

$$V_{ij} = \frac{Y_{ij} + \overline{M}_j X_{ij}}{(Y_{jj} + \overline{M}_j X_{jj}) + \overline{M}_j^{-1} - \text{Tr} Y} V_{jj} \quad (i, j = 1, 2, 3 \text{ and } i \neq j), \quad (13)$$

where

$$F = \left(\frac{m_{1D}^2 m_{2D}^2 m_{3D}^2}{m_1 m_2 m_3} \right)^{\frac{1}{3}}, \quad (14)$$

and $\overline{M}_i = e^{-2\eta_3 \lambda_3 - 2\sqrt{3}\eta_8 \lambda_8}$ which satisfy

$$\overline{M}_1^{-1} + \overline{M}_2^{-1} + \overline{M}_3^{-1} = X_{11} + X_{22} + X_{33} \equiv A, \quad (15)$$

$$\overline{M}_1 + \overline{M}_2 + \overline{M}_3 = Y_{11} + Y_{22} + Y_{33} \equiv B. \quad (16)$$

Here $\lambda_{3,8}$ are diagonal Gell-Mann matrices. The elements of X and Y can be expressed as

$$X_{ij} = \frac{1}{\overline{m}_{iD} \overline{m}_{jD}} \sum_{k=1}^3 \overline{m}_{\nu k} S_{ik} S_{jk}, \quad (17)$$

$$Y_{ij} = \overline{m}_{iD} \overline{m}_{jD} \sum_{k=1}^3 \frac{1}{\overline{m}_{\nu k}} S_{ik} S_{jk}, \quad (18)$$

where $\overline{m}_{iD} = e^{-\xi_3 \lambda_3 - \sqrt{3}\xi_8 \lambda_8}$ and $\overline{m}_{\nu k} = e^{-2\kappa_3 \lambda_3 - 2\sqrt{3}\kappa_8 \lambda_8}$.

When $m_0 = m_1 \approx m_2 \approx m_3$, we define δ_{12} and δ_{13} as

$$m_2 = m_0 (1 + \delta_{12}), \quad m_3 = m_0 (1 + \delta_{13}) \quad (19)$$

which satisfy $\delta_{12} \ll \delta_{13}$. From the above relations it is easy for one to obtain

$$M_i \approx \frac{m_{iD}^2}{m_0}, \quad i = 1, 2, 3 \quad (20)$$

and

$$\beta_{ij} \approx -\frac{m_{iD}}{m_{jD}} \left(H_{ij}(\delta_{12}, \delta_{13}) - g H_{ij} \left(\frac{\delta_{12}}{1 + \delta_{12}}, \frac{\delta_{13}}{1 + \delta_{13}} \right) \right), \quad 1 \leq i < j \leq 3, \quad (21)$$

where the function H_{ij} is defined as

$$H_{ij}(\delta_{12}, \delta_{13}) = \delta_{12} S_{i2} S_{j2} + \delta_{13} S_{i3} S_{j3} \quad (22)$$

and

$$g = \begin{cases} \frac{m_{2D}^2}{m_{3D}^2}, & \text{for } \beta_{12} \text{ and } \beta_{13} \\ \frac{m_{1D}^2}{m_{3D}^2}, & \text{for } \beta_{23} \end{cases}. \quad (23)$$

We can see the RH mixing angles are small. However their scales are sensitive to the inputs of S_{12} , S_{13} , δ_{12} and δ_{13} . For example, $\beta_{13} \approx \frac{m_{1D}}{m_{3D}} \delta_{12} S_{12} S_{32}$ when $S_{13} = 0$ and $\beta_{13} \rightarrow 0$ when $\frac{S_{13}}{S_{32}} \approx \frac{\delta_{12}}{\delta_{13}}$. Despite this sensitivity, we find

$$\beta_{12} \approx -\frac{m_{1D}}{m_{2D}} \delta_{12} S_{12} S_{22}, \quad \beta_{13} \approx \frac{m_{1D}}{m_{3D}} \delta_{12} S_{12} S_{32}, \quad \beta_{23} \approx \frac{m_{2D}}{m_{3D}} \delta_{13} S_{23} S_{33} \quad (24)$$

when $S_{13} = 0$ and for SMA

$$\beta_{12} \approx -\frac{m_{1D}}{m_{2D}} \delta_{13} S_{13} S_{23}, \quad \beta_{13} \approx \frac{m_{1D}}{m_{3D}} \delta_{13} S_{13} S_{33}, \quad \beta_{23} \approx \frac{m_{2D}}{m_{3D}} \delta_{13} S_{23} S_{33} \quad (25)$$

when $S_{13} \gtrsim S_{12}$.

If assuming $m_0 \sim 1\text{eV}$, one has $M_1 = 1.6 \times 10^3 \text{ GeV}$, $M_2 = 1.8 \times 10^8 \text{ GeV}$ and $M_3 = 1.2 \times 10^{13} \text{ GeV}$. Note that $M_{2,3}$ are in the intermediate scale while M_1 , so suprise, is in the electric-weak scale.

Assuming $D_L = I$, all the corresponding relations can be obtained by replace S with U .

III. HIERARCHICAL SPECTRUM

In this case the two heavier neutrino masses can be written as $m_2 \approx \sqrt{\Delta m_{21}^2}$ and $m_3 \approx \sqrt{\Delta m_{31}^2}$. Setting $\theta_{23}^\nu = \frac{\pi}{4}$, the neutrino mixing matrix U have the form

$$U = \begin{pmatrix} \cos \theta_{12}^\nu \cos \theta_{13}^\nu & \sin \theta_{12}^\nu \cos \theta_{13}^\nu & \sin \theta_{13}^\nu \\ -\frac{\sin \theta_{12}^\nu + \cos \theta_{12}^\nu \sin \theta_{13}^\nu}{\sqrt{2}} & \frac{\cos \theta_{12}^\nu - \sin \theta_{12}^\nu \sin \theta_{13}^\nu}{\sqrt{2}} & \frac{\cos \theta_{13}^\nu}{\sqrt{2}} \\ \frac{\sin \theta_{12}^\nu - \cos \theta_{12}^\nu \sin \theta_{13}^\nu}{\sqrt{2}} & -\frac{\cos \theta_{12}^\nu + \sin \theta_{12}^\nu \sin \theta_{13}^\nu}{\sqrt{2}} & \frac{\cos \theta_{13}^\nu}{\sqrt{2}} \end{pmatrix}. \quad (26)$$

In follows, we first consider the small mixing solution to the solar neutrino problem and then VO, LMA and LOW are embodied in a unitized framework, i.e. the large θ_{12}^ν .

A. small θ_{12}^ν (SMA)

We will always assume $\sin^2 2\theta_{12}^\nu \gtrsim 10^{-3}$ so that it remain in the SMA region. When θ_{12}^ν is small there are three possibilities by comparing θ_{13}^ν with θ_{12}^ν .

$$1. \theta_{12}^\nu \gg \theta_{13}^\nu$$

We can know from Ref. [5] that in this case

$$M_1 \approx f_1 \frac{m_{1D}^2}{m_2} \frac{1}{\sin^2 \theta_{12}^\nu}, \quad M_2 \approx 2 \frac{m_{2D}^2}{m_3}, \quad M_3 \approx \frac{1}{2} f_1^{-1} \frac{m_{3D}^2}{m_1} \sin^2 \theta_{12}^\nu \quad (27)$$

and

$$\beta_{12} \approx -\frac{1}{\sqrt{2}} f_1 \frac{m_{1D}}{m_{2D}} \cot \theta_{12}^\nu, \quad \beta_{13} \approx \sqrt{2} f_1 \frac{m_{1D}}{m_{3D}} \cot \theta_{12}^\nu, \quad \beta_{23} \approx -\frac{m_{2D}}{m_{3D}} \quad (28)$$

where $f = \frac{r_{21}}{r_{21} + \cot^2 \theta_{12}}$ and $r_{21} = \frac{m_2}{m_1} \gg 1$. The relations are obtained when $\theta_{13}^\nu \rightarrow 0$ so that

$$U = \begin{pmatrix} \cos \theta_{12}^\nu & \sin \theta_{12}^\nu & 0 \\ -\frac{\sin \theta_{12}^\nu}{\sqrt{2}} & \frac{\cos \theta_{12}^\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin \theta_{12}^\nu}{\sqrt{2}} & -\frac{\cos \theta_{12}^\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (29)$$

Note that, unlike what one would expect when no mixing occurs $M_i \propto m_i^{-1}$, M_i have rotate dependence on m_i^{-1} in the sense: $M_i \propto m_{i1}^{-1}$ where we define

$$i_k \equiv (i + k) \bmod 3 \quad i, k = 1, 2, 3. \quad (30)$$

As noted in Ref. [5] the RH mixing angles scale linearly with the ratios of the Dirac neutrino masses

$$\beta_{ij} \sim \frac{m_{iD}}{m_{jD}}, \quad 1 \leq i < j \leq 3. \quad (31)$$

which is different with the LH quark mixing angles where one obtains $\tan \theta^{\text{quark}} \approx \sqrt{\frac{m_d}{m_s}}$ in two-generation case [14].

$$2. \theta_{12}^\nu \ll \theta_{13}^\nu$$

In this case

$$U \approx \begin{pmatrix} 1 & \sin \theta_{12}^\nu & \sin \theta_{13}^\nu \\ -\frac{\sin \theta_{13}^\nu}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sin \theta_{13}^\nu}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (32)$$

Here U is not a strict orthogonal matrix as required. However it brings no trouble in since only the relative magnitudes of the elements of U are used in our analysis. We have

$$M_1 \approx \frac{m_{1D}^2}{m_3} \frac{1}{\sin^2 \theta_{13}^\nu}, \quad M_2 \approx 2 \frac{m_{2D}^2}{m_1} \frac{\sin^2 \theta_{13}^\nu}{r_{21} \sin^2 \theta_{13}^\nu + 1}, \quad M_3 \approx \frac{1}{2} \frac{m_{3D}^2}{m_2} (r_{21} \sin^2 \theta_{13}^\nu + 1) \quad (33)$$

and

$$\beta_{12} \approx -\frac{m_{1D}}{m_{2D}} \frac{1}{\sqrt{2} \sin \theta_{13}^\nu}, \quad \beta_{13} \approx \sqrt{2} \frac{m_{1D}}{m_{3D}} \sin \theta_{13}^\nu \frac{r_{21}}{r_{21} \sin^2 \theta_{13}^\nu + 1}, \quad \beta_{23} \approx \frac{m_{2D}}{m_{3D}}. \quad (34)$$

From Eq. (33) one has $M_i \propto m_{i2}^{-1}$ when $r_{21} \lesssim \frac{U_{\tau 2}^2}{U_{\tau 1}^2} \approx \sin^{-1} \theta_{13}^\nu$ and $M_{1(3)} \propto m_{3(1)}^{-1}$ and $M_2 \propto m_2^{-1}$ when $r_{21} \gg \sin^{-1} \theta_{13}^\nu$.

3. $\theta_{12}^\nu \sim \theta_{13}^\nu$

For convenient, we set $U_{\tau 1} = 0$, that is $\sin \theta_{13}^\nu = \tan \theta_{12}^\nu$. So that

$$U = \begin{pmatrix} \sqrt{\cos 2\theta_{12}^\nu} & \sqrt{\cos 2\theta_{12}^\nu} \tan \theta_{12}^\nu & \tan \theta_{12}^\nu \\ -\sqrt{2} \sin \theta_{12}^\nu & \frac{\cos 2\theta_{12}^\nu}{\sqrt{2} \cos \theta_{12}^\nu} & \frac{\sqrt{\cos 2\theta_{12}^\nu}}{\sqrt{2} \cos \theta_{12}^\nu} \\ 0 & -\frac{1}{\sqrt{2} \cos \theta_{12}^\nu} & \frac{\sqrt{\cos 2\theta_{12}^\nu}}{\sqrt{2} \cos \theta_{12}^\nu} \end{pmatrix}. \quad (35)$$

We find

$$M_1 \approx \frac{m_{1D}^2}{m_3} \cot^2 \theta_{12}^\nu, \quad (36a)$$

$$M_2 \approx \begin{cases} 2 \frac{m_{2D}^2}{m_1} \sin^2 \theta_{12}^\nu, & \text{if } r_{21} < r_{21}^{\text{res}} \\ \frac{1}{2} \frac{m_{3D}^2}{m_2}, & \text{if } r_{21} > r_{21}^{\text{res}} \end{cases} \quad (36b)$$

$$M_3 \approx \begin{cases} \frac{1}{2} \frac{m_{3D}^2}{m_2}, & \text{if } r_{21} < r_{21}^{\text{res}} \\ 2 \frac{m_{2D}^2}{m_1} \sin^2 \theta_{12}^\nu, & \text{if } r_{21} > r_{21}^{\text{res}} \end{cases} \quad (36c)$$

where $r_{21}^{\text{res}} = \frac{1}{4} \frac{m_{3D}^2}{m_{2D}^2} \csc^2 \theta_{12}^\nu$. We have two degenerate masses $M_2 = M_3$ when $r_{21} = r_{21}^{\text{res}}$.

The second RH mixing angle β_{23} is given in

$$\tan 2\beta_{23} \approx \frac{\frac{1}{2} \frac{m_{3D}^2}{m_{2D}^2} \csc^2 \theta_{12}^\nu}{e^{4\kappa_3} - \frac{1}{4} \frac{m_{3D}^2}{m_{2D}^2} \csc^2 \theta_{12}^\nu} \approx -\frac{2m_{2D}/m_{3D}}{1 - r_{21}/r_{21}^{\text{res}}} \quad (37)$$

or

$$\sin 2\beta_{23} \approx -\frac{2m_{2D}/m_{3D}}{\sqrt{(1 - r_{21}/r_{21}^{\text{res}})^2 + 4m_{2D}^2/m_{3D}^2}}, \quad (38)$$

where $\kappa_3 = \frac{1}{4} \ln \frac{m_2}{m_2}$ [5]. The other two RH mixing angles are both small and can be expressed in β_{23} as follows

$$\beta_{12} \approx \frac{1}{\sqrt{2} \sin \theta_{12}^\nu} \left(-\frac{m_{1D}}{m_{2D}} \cos \beta_{23} + \frac{m_{1D}}{m_{3D}} \sin \beta_{23} \right), \quad (39)$$

$$\beta_{13} \approx \frac{1}{\sqrt{2} \sin \theta_{12}^\nu} \left(-\frac{m_{1D}}{m_{2D}} \cos \beta_{23} + \frac{m_{1D}}{m_{3D}} \sin \beta_{23} \right). \quad (40)$$

The behaviors of M_i , $\eta_3 \left(= \frac{1}{4} \ln \frac{M_2}{M_1} \right)$, $\eta_8 \left(= \frac{1}{12} \ln \frac{M_3^2}{M_1 M_2} \right)$ and β_{ij} as functions of κ_3 are shown in Fig. 1. In Fig. 2 we have plotted M_2 , M_3 and $\sin^2 2\beta_{23}$ near κ_3^{res} where κ_3^{res} is the location of the resonance defined as

$$e^{4\kappa_3^{\text{res}}} \equiv \frac{1}{4} \frac{m_{3D}^2}{m_{2D}^2} \csc^2 \theta_{12}^\nu. \quad (41)$$

The behavior of $\sin^2 2\beta_{23}$ as a function of κ_3 is clearly that of a resonance peaked at $\kappa_3 = \kappa_3^{\text{res}}$, when $\sin^2 2\beta_{23} = 1$. We can define the resonance width δ_{κ_3} as that of κ_3 around κ_3^{res} for which $\sin^2 2\beta_{23}$ becomes $\frac{1}{2}$ instead of the maximum value, unity. It is given by

$$\delta_{\kappa_3} \approx \frac{m_{2D}}{m_{3D}}. \quad (42)$$

The situation is very like that in the matter-enhanced $\nu_e \leftrightarrow \nu_\mu$ oscillation in the sun while here r_{21} plays a part of the effective potential $V = 2\sqrt{2}G_f N_e E_\nu$. Here G_f is the Fermi constant, N_e is the electron number density of the matter and E_ν is the neutrino energy. When $\sin^2 2\beta_{23} = 1$ (that is when $\kappa_3^{\text{res}} \approx 4.2$ i.e. $m_1 \approx 1.2 \times 10^{-10}$ eV), substituting the SMA data in Eq. (5) into Eq. (36) we have

$$M_1 \approx 1.4 \times 10^7 \text{ GeV}, \quad M_2 \approx M_3 \approx 5 \times 10^{15} \text{ GeV}. \quad (43)$$

B. large θ_{12}^ν

In this case it is no need to consider the relative magnitude of θ_{12}^ν and θ_{13}^ν since one always has $U_{\tau 1} \approx \frac{\sin \theta_{12}^\nu}{\sqrt{2}}$. We shall therefore consider this problem in the following two possibilities according to the magnitude of θ_{13}^ν .

1. θ_{13}^ν is tiny

We find, when

$$\sin^2 \theta_{13}^\nu \ll \min \left(\frac{m_2}{m_3} U_{12}^2, \quad \frac{m_{1D}^2}{m_{2D}^2} U_{\tau 3}^2 \right) \approx \frac{1}{2} \frac{m_{1D}^2}{m_{2D}^2}, \quad (44)$$

the RH Majorana masses are

$$M_1 \approx \begin{cases} \frac{m_{1D}^2}{m_2} \frac{1}{\sin^2 \theta_{12}^\nu} & \frac{m_3}{m_2} < 2 \frac{m_{2D}^2}{m_{1D}^2} \sin^2 \theta_{12}^\nu \\ 2 \frac{m_{2D}^2}{m_3} & \frac{m_3}{m_2} > 2 \frac{m_{2D}^2}{m_{1D}^2} \sin^2 \theta_{12}^\nu \end{cases} \quad (45a)$$

$$M_2 \approx \begin{cases} 2 \frac{m_{2D}^2}{m_3} & \frac{m_3}{m_2} < 2 \frac{m_{2D}^2}{m_{1D}^2} \sin^2 \theta_{12}^\nu \\ \frac{m_{1D}^2}{m_2} \frac{1}{\sin^2 \theta_{12}^\nu} & \frac{m_3}{m_2} > 2 \frac{m_{2D}^2}{m_{1D}^2} \sin^2 \theta_{12}^\nu \end{cases} \quad (45b)$$

$$M_3 \approx \frac{1}{2} \frac{m_{3D}^2}{m_1} \sin^2 \theta_{12}^\nu \quad (45c)$$

and the RH mixing angles

$$\beta_{13} \approx \sqrt{2} \frac{m_{1D}}{m_{3D}} \cot \theta_{12}^\nu, \quad \beta_{23} \approx -\frac{m_{2D}}{m_{3D}}. \quad (46)$$

We also give a numerical result for β_{12} together with M_i , $\eta_{3,8}$ and the other two mixing angles as functions of $\kappa_8 \left(= \frac{1}{12} \ln \frac{m_3^2}{m_1 m_2} \right)$ in Fig. 3 taking $m_3^2 = 0.1 \text{ eV}^2$ and $\kappa_3 = 2$. We also plotted M_1 , M_2 and $\sin^2 2\beta_{12}$ near $r_{32}^{\text{res}} \approx 2 \frac{m_{2D}^2}{m_{1D}^2} \sin^2 \theta_{12}^\nu$ in Fig. 4.

2. θ_{13}^ν is not so small

We find, when θ_{13}^ν is small and satisfies

$$\sin^2 \theta_{13}^\nu \gtrsim \max \left(\frac{m_2}{m_3} U_{12}^2, \quad \frac{m_{1D}^2}{m_{2D}^2} U_{\tau 3}^2 \right) \approx \frac{m_2}{m_3} \sin^2 \theta_{12}^\nu, \quad (47)$$

the RH Majorana neutrino masses are hierarchical

$$M_1 \approx f_1^{-1} \frac{m_D^2}{m_2}, \quad M_2 \approx f_1 \frac{m_{2D}^2}{m_3} \frac{1}{U_{\tau 1}^2}, \quad M_3 \approx \frac{m_{3D}^2}{m_1} U_{\tau 1}^2, \quad (48)$$

where $f_1 = \left(U_{e2}^2 + \frac{m_3}{m_2} U_{e3}^2 + \frac{m_{1D}^2 m_3}{m_{2D}^2 m_2} U_{\mu 3}^2 \right)$ and all the three RH mixing angles are small

$$\beta_{12} \approx -\frac{U_{e2} U_{\mu 2} + \frac{m_3}{m_2} U_{e3} U_{\mu 3}}{S_{e2}^2 + \frac{m_3}{m_2} S_{\tau 3}^2} \frac{m_{1D}}{m_{2D}}, \quad \beta_{13} \approx \frac{m_{1D}}{m_{3D}} \frac{U_{e1}}{U_{\tau 1}}, \quad \beta_{23} \approx \frac{m_{2D}}{m_{3D}} \frac{U_{\mu 1}}{U_{\tau 1}}. \quad (49)$$

Here we also have rotate dependence of M_i on m_i^{-1} and the RH mixing angles β_{ij} scale linearly with the ratios of the Dirac neutrino masses.

IV. SUMMARY AND DISCUSSION

Separating the solution regions of the solar neutrino problem in two cases according to the value of θ_{12}^ν , we have derived simple relations between parameters of the RH and LH Majorana neutrino masses and mixing in the context of the seesaw mechanism and quark-lepton symmetry within the framework of three families. Especially, as an extension of our previous work, we have embodied quasidegenerate light neutrino mass case and the influence of nonzero U_{e3} on the properties (masses and mixing) of the RH Majorana neutrinos. The CP-violating effect has not included. We find

1. quasidegenerate neutrino spectrum leads to hierarchical RH Majorana masses and small RH mixing angles which scale linearly with the ratios of the Dirac masses

$$\beta_{ij} \sim \frac{m_{iD}}{m_{jD}}, \quad 1 \leq i < j \leq 3. \quad (50)$$

2. For SMA, resonance like behavior of $\sin^2 2\beta_{23}$ is found when $U_{\tau 1} \approx 0$. We find when $m_1 \approx 1.2 \times 10^{-10}$ eV one has $\sin^2 2\beta_{23} = 1$, $M_1 \approx 1.4 \times 10^7$ GeV and $M_2 \approx M_3 \approx 5 \times 10^{15}$ GeV. M_2, M_3 are near the scale of GUT and $M_3 \approx 5 \times 10^{15}$ GeV for a wide range of r_{21} , quantitatively, for r_{21} below r_{21}^{res} .
3. The behavior $\sin^2 2\beta_{12}$ as a functions of κ_8 if $\frac{m_2}{m_1}$ is given) is a resonance peaked at $r_{32} = r_{32}^{\text{res}}$ for large θ_{12}^ν case while θ_{13}^ν is tiny and one has two lighter degenerate RH Majorana masses at this point. In Ref. [5] we have not discuss this case considering that m_2 will far less than the lower bound of the solar neutrino solutions if taking $m_3^2 \sim 10^{-3}$ eV² which is around the best-fit value. However, the degenerate M_1 and M_2 could be coincident with the experimental results considering that m_3^2 can reach to about 10^{-1} eV² and then one has $m_2^2 \sim 10^{-11}$ eV² which lies in the region of the vacuum explanation to the solar neutrino anomaly but still far less than the lower bound of the LMA and LOW solutions. For LMA and LOW, one always has $\frac{m_3}{m_2} < 2 \frac{m_{3D}^2}{m_{1D}^2} \sin^2 \theta_{12}^\nu$ and so

$$M_1 \approx \frac{m_{1D}^2}{m_2} \frac{1}{\sin^2 \theta_{12}^\nu}, \quad M_2 \approx 2 \frac{m_{2D}^2}{m_3}, \quad M_3 \approx \frac{1}{2} \frac{m_{3D}^2}{m_1} \sin^2 \theta_{12}^\nu; \quad (51)$$

$$\beta_{12} \approx -\frac{1}{\sqrt{2}} \frac{m_{1D}}{m_{2D}} \cot \theta_{12}^\nu, \quad \beta_{13} \approx \sqrt{2} \frac{m_{1D}}{m_{3D}} \cot \theta_{12}^\nu, \quad \beta_{23} \approx -\frac{m_{2D}}{m_{3D}}, \quad (52)$$

that is, $M_i \propto m_{i1}^{-1}$ and $\beta_{ij} \sim \frac{m_{iD}}{m_{jD}}$ ($1 \leq i < j \leq 3$). For VO, the corresponding RH masses when $r_{32} = r_{32}^{\text{res}}$ are $M_1 \approx M_2 \approx 1.2 \times 10^9$ GeV and $M_3 \approx 1.8 \times 10^{17} r_{21}$ GeV.

4. An interesting analogue of the RH parameters around the resonance with the behavior of the matter-enhanced neutrino conversion is presented.
5. As an accessory consequence, we present a numerical method for calculating the physical parameters in the seesaw mechanism which usually involving extreme large and extreme small quantities simultaneously. In our numerical results, eigenvalues larger than unit and the corresponding eigenvectors of X are directly obtained from X . By solving the eigenequation of Y , we obtain the inverse of the eigenvalues of X larger than unit and the corresponding eigenvectors. The validness of this method can be verified simply by the condition that the product of the three eigenvalues of X is unit. We find this condition is satisfied well.

The results are dependent on the precise determination of U_{e3} . Such a goal is expected to be reached in the neutrino long baseline experiment, registration of the neutrino bursts from the Galactic supernova by existing detectors SK and SNO, and the neutrino factories [15]. In this paper we have not discuss the case when M_R and m_D are complex. We hope return to it in future.

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FIGURES

FIG. 1. The behavior of the RH Majorana masses, mass ratios and the RH mixing angles as functions of κ_3 for the SMA solution to the solar neutrino anomaly when $U_{\tau 1} = 0$. We take $m_2^2 = 5.4 \times 10^{-6} \text{ eV}^2$, $m_3^2 = 5.9 \times 10^{-3} \text{ eV}^2$, $\sin^2 2\theta_{12}' = 6.0 \times 10^{-3}$, $\theta_{13}' = 0.0$ and $\sin^2 2\theta_{23}' = 1.0$.

FIG. 2. The behavior of M_2 , M_3 and $\sin^2 2\beta_{23}$ near κ_3^{res} for the SMA solution. The values of the parameters are the same as in Fig. 1.

FIG. 3. The behavior of the RH Majorana masses, mass ratios and the RH mixing angles as functions of κ_8 for the vacuum oscillation solution to the solar neutrino anomaly when $U_{e3} = 0$. We take $m_2^2 = 6.5 \times 10^{-11} \text{ eV}^2$, $m_3^2 = 0.1 \text{ eV}^2$, $\sin^2 2\theta_{12}' = 0.75$ and $\sin^2 2\theta_{23}' = 1.0$.

FIG. 4. The behavior of M_1 , M_2 and $\sin^2 2\beta_{12}$ near κ_8^{res} for the vacuum oscillation solution when $U_{e3} = 0$. The values of the parameters are the same as in Fig. 3.